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SUFFICIENT CONDITIONS FOR AN EXTREMUM IN EIGENVALUE OPTIMIZATION PROBLEMS*

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Problems of maximizing the minimum eigenvalue of selfadjoint matrices and differential operators are considered. These problems arise when optimizing the critical buckling force or the fundamental frequency of the natrual vibrations of elastic structures /1—4/. It has been shown /5— 10/ that the extremal eigenvalue turns out to be multiple in a number of cases. The multiplicity of the critical load in maximization problems for the critical buckling force denotes the presence of several buckling modes for this load.

Sufficient conditions for a local extremum for single and double eigenvalues are obtained for discrete and continuous systems. In the continuous case, the sufficient conditions for an extremum are derived using the example of a rod buckling problem. The conditions obtained are constructive in nature and can be utilized in different eigenvalue optimization problems.

1. Consider the generalized eigenvalue problem

$$A(h)u = \lambda B(h)u \tag{1.1}$$

Here $A(h)$ and $B(h)$ are positive-definite symmetric $m \times m$ matrices with coefficients $a_{ij}(h)$ and $b_{ij}(h)$, respectively, that depend continuously on the components of the parameter vector h of dimensions n, while u is a vector of dimensions m, and λ is the eigenvalue.

Problem (1.1) has a complete system of eigenvectors u^i $(i = 1, 2, \ldots, m)$ and the eigenvalue sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$, where we assume ($\delta_{ij}$ is the Kronecker delta)

$$(u^i, \ B(h)u^j) = \delta_{ij} \tag{1.2}$$

Here and henceforth, parentheses denote the scalar product of vectors.

We pose the following problem: it is required to find the parameter vector $h = (h_1, \ldots, h_n)$ for which the minimum eigennumber λ_1 of problem (1.1) reaches a maximum value under the condition

$$F(h) = 0 \tag{1.3}$$

where $F(h)$ is a certain fixed linear function of the vector argument h.

Let λ_1 and u^i $(i = 1, 2, \ldots, m)$ be the eigennumbers and eigenvectors of problem (1.1) calculated for a certain h. We shall first assume that λ_1 is a single eigenvalue. We will apply the result of analytic perturbation of the symmetric operator spectrum /11/. We give the vector h an increment in the form of the vector εk, $k = (k_1, \ldots, k_n)$, where ε is a small positive number. It follows from (1.3) that the vector k should satisfy the condition

$$(f^\circ, \ k) = 0, \quad f^\circ = \nabla F \tag{1.4}$$

where f° is a fixed vector giving the gradient of the function $F(h)$. As a result of perturbation of the parameter vector, the eignnumber λ_1 and the eigenvector u^1 receive increments which can be written in the form

$$u = u^1 + \varepsilon v^1 + \varepsilon^2 v^2 + o(\varepsilon^2), \quad \lambda = \lambda_1 + \varepsilon\mu + \varepsilon^2\eta + o(\varepsilon^2)$$

Substituting the expansions obtained into (1.1) and collecting terms of the zeroth, first, and second powers of ε, we obtain

$$A(h)u^1 = \lambda_1 B(h)u^1 \tag{1.5}$$

$$A_1(h, \ k)u^1 + A(h)v^1 = \lambda_1 B(h)v^1 + \mu B(h)u^1 + \lambda_1 B_1(h, k)u^1 \tag{1.6}$$

$$A_2(h, \ k)u^1 + A_1(h, \ k)v^1 + A(h)v^2 = \lambda_1 B(h)v^2 + \mu B(h)v^1 + \mu B_1(h, \ k)u^1 + \lambda_1 B_1(h, \ k)v^1 + \lambda_1 B_2(h, \ k)u^1 + \eta B(h)u^1 \tag{1.7}$$

Here $A_1(h, k)$, $B_1(h, k)$ are matrices with the components $(\nabla a_{ij}, \ k)$ and $(\nabla b_{ij}, \ k)$ $(i, j = 1, 2, \ldots, m)$, respectively, where

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$$\nabla a_{ij} = \left(\frac{\partial a_{ij}}{\partial h_1}, \ldots, \frac{\partial a_{ij}}{\partial h_n}\right)(h), \quad \nabla b_{ij} = \left(\frac{\partial b_{ij}}{\partial h_1}, \ldots, \frac{\partial b_{ij}}{\partial h_n}\right)(h)$$

$A_2(h, k)$ and $B_2(h, k)$ are matrices with the components

$$\frac{1}{2}\sum_{s,\,t=1}^{n} \frac{\partial^2 a_{ij}}{\partial h_s \partial h_t}(h) k_s k_t, \quad \frac{1}{2}\sum_{s,\,t=1}^{n} \frac{\partial^2 b_{ij}}{\partial h_s \partial h_t}(h) k_s k_t, \quad i, j = 1, 2, \ldots, m$$

It is convenient to introduce the following notation (f^l are vectors of dimensions n)

$$C(h) = A(h) - \lambda_1 B(h) \tag{1.8}$$
$$C_i(h, k) = A_i(h, k) - \lambda_1 B_i(h, k), \quad i = 1, 2$$
$$f^l = \sum_{i,\,j=1}^{m} u_i{}^1 u_j{}^l (\nabla a_{ij} - \lambda_1 \nabla b_{ij})(h), \quad l = 1, \ldots, m$$

Here u_j^l are components of the eigenvectors u^l, $j, l = 1, 2, \ldots, m$. We note that the matrices C_i, A_i, B_i $(i = 1, 2)$, and C are symmetric because of the symmetry of the matrices A and B.

Multiplying (1.6) scalarly by the vector u^1, using the symmetry of the matrices $A(h)$ and $B(h)$, conditions (1.2) and (1.5), we obtain

$$\mu = (C_1(h, k)u^1, u^1) = (f^1, k)$$

If h is a vector realizing the solution of the above optimization problem, it is necessary that for any vector $k = (k_1, \ldots, k_n)$, the equation $\mu = 0$ should be satisfied for $(f^\circ, k) = 0$. Hence we obtain

$$f^1 = df^\circ \tag{1.9}$$

with a certain constant d which yields the necessary extremum condition in the problem under consideration for maximizing the least eigenvalue λ_1.

We assume that condition (1.9) is satisfied, and, using the equation $\mu = 0$, we can write (1.6) in the form

$$C(h) v^1 = -C_1(h, k) u^1 \tag{1.10}$$

where the matrix C_1 is defined by (1.8).

The vector v^1 can be represented in the form of a linear combination of the vectors u^l $(l = 1, \ldots, m)$, i.e.

$$v^1 = c_1 u^1 + \ldots + c_m u^m \tag{1.11}$$

Substituting this expansion into (1.10) may successively multiplying it scalarly by the vector u^l, taking account of (1.5) and the notation (1.8), we obtain

$$v^1 = c_1 u^1 - \sum_{l=2}^{m} \frac{(f^l, k)}{\lambda_l - \lambda_1} u^l \tag{1.12}$$

The constant c_1 is determined from the normalization condition and does not influence the subsequent calculations.

We multiply (1.7) scalarly by the vector u^1 by using (1.5), the equation $\mu = 0$ and condition (1.2). Finally, we obtain an expression for the second correction to the eigenvalue

$$\eta = (C_2(h, k)u^1, u^1) + (C_1(h, k)u^1, v^1)$$

where C_1 and C_2 are defined in (1.8). We introduce the notation

$$d_{st} = \frac{1}{2}\sum_{i,\,j=1}^{m} u_i{}^1 u_j{}^1 \left(\frac{\partial^2 a_{ij}}{\partial h_s \partial h_t} - \lambda_1 \frac{\partial^2 b_{ij}}{\partial h_s \partial h_t}\right)(h)$$

and denote by $D(h)$ the matrix with components d_{st}; $s, t = 1, \ldots, n$. From (1.12), the stationarity condition $(f^1, k) = 0$, and the notation (1.8) introduced above, we have

$$\eta = (D(h)k, k) - \sum_{l=2}^{m} \frac{(f^l, k)^2}{\lambda_l - \lambda_1} \tag{1.13}$$

Expression (1.13) determines the magnitude of the second variation of the eigenvalue λ_1 for values of the parameter vector h satisfying the stationarity condition (1.9).

Assertion 1. The sufficient condition for a local extremum of the stationary value of the vector h is expressed by the inequality $\eta(h, k) < 0$ for any variations $k = (k_1, \ldots, k_n)$ satisfying (1.4).

This condition is equivalent to the condition of negative definiteness of the form (1.13) considered as a quadratic form of the components of the vector k on the hyperplane $(f^\circ, k) = 0$.

This condition of negative definiteness of the matrix $D(h)$ is obviously sufficient for the optimality of the stationary vector h, since the second term in (1.13) is always non-positive because $\lambda_1 < \lambda_l$ $(l = 2,\ 3,\ \ldots,\ m)$.

We will examine the special case when the dependence of the matrices A and B of problem (1.1) on the vector component h is linear, and hence $D \equiv 0$. If the rank r of the vector system $\{f^l\}$ $(l = 1, 2, \ldots, m)$ equals the dimensions of the vector h, i.e., $r = n$ (which is possible for $m \geqslant n$), then $\eta < 0$ for any non-zero vectors k. Indeed, in this case $\eta \leqslant 0$ and $\eta = 0$ only for $(f^l, k) = 0$ $(l = 1, 2, \ldots, m)$. Because $r = n$, it hence follows that $k \equiv 0$. We note that because of conditions (1.4) and (1.9), the vector k should satisfy the condition $(f^1, k) = 0$. If $r < n$, then non-zero vectors k always exist such that $(f^l, k) = 0$ $(l = 1, 2, \ldots, m)$. In this case the question of an extremum is solved by including higher-order variations. Therefore, for a linear dependence of the matrices A and B on the vector components h, the condition $r = n$ is sufficient for the optimality of the stationary vector h.

2. We will examine the case when the double eigenvalue $\lambda_1 = \lambda_2 < \lambda_3 \leqslant \lambda_4 \ldots$ corresponds to the vector h realizing the solution of the problem of maximizing the minimum eigenvalue of problem (1.1). As in Sect.1, it is assumed that the orthogonal eigenvectors u^i $(i = 1, 2, \ldots, m)$ normalized in the sense of (1.2) correspond to the eigenvalues λ_1. Any linear combination of the vectors u^1 and u^2

$$u^\circ = \gamma_1 u^1 + \gamma_2 u^2, \quad \gamma_1^2 + \gamma_2^2 \neq 0 \tag{2.1}$$

is also an eigenvector corresponding to the double value $\lambda_1 = \lambda_2$.

We give the vector h an increment εk, where ε is a small positive number, and we calculate the increment of the eigenvalue λ_i. Using the expansion $\lambda = \lambda_1 + \varepsilon\mu + \varepsilon^2\eta + o(\varepsilon^2)$ and $u = u^\circ + \varepsilon v^1 + \varepsilon^2 v^2 + o(\varepsilon^2)$, in this case /11/, we arrive at (1.5)–(1.7) with the sole difference that instead of u^1 we will have $u^\circ = \gamma_1 u^1 + \gamma_2 u^2$. The constants γ_1 and γ_2 are also to be determined form the equations of the perturbation method.

Multiplying (1.6) scalarly by u^1 and u^2, we obtain a system of linear homogeneous equations in γ_1 and γ_2. Equating the determinant of this system to zero, we arrive at a quadratic equation to determine μ

$$\mu^2 - (\alpha_{11} + \alpha_{22})\,\mu + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) = 0 \tag{2.2}$$
$$\alpha_{ij} = (C_1(h,\ k)\,u^i,\ u^j), \quad i,\ j = 1,\ 2$$

where the matrix $C_1(h, k)$ is determined by the second equation in (1.8). Because of the symmetry in the matrix C_1 the coefficient $\alpha_{12} = \alpha_{21}$, which ensures that the roots of (2.2) will be real. We introduce the n-dimensional vectors

$$f_s^{\,l} = \sum_{i,\,j=1}^{m} u_i^{\,l} u_j^{\,s} (\nabla a_{ij} - \lambda_1 \nabla b_{ij})(h), \quad l, s = 1, \ldots, m \tag{2.3}$$

We note that $f_s^{\,l} = f_l^{\,s}$ because of the symmetry of a_{ij}, b_{ij}. Taking this and the notation (1.8) into account, we write α_{ij} in the form

$$\alpha_{ij} = (f_j^{\,i},\ k), \quad i,\ j = 1,\ 2 \tag{2.4}$$

If the vector h realizes the solution of the optimization problem, then $\mu_1\mu_2 \leqslant 0$ is necessary, where μ_1 and μ_2 are the roots of the quadratic equation (2.2). This means that the minimum eigenvalue λ_1 is maximum, and allowable variations of εk do not result in its enlargement /12/.

Using (2.2) and (2.4) we give the condition $\mu_1\mu_2 \leqslant 0$ the form

$$L(h,\ k) = (f_1^{\,2},\ k)^2 - (f_1^{\,1},\ k)(f_2^{\,2},\ k) \geqslant 0 \tag{2.5}$$

for any k satisfying the isoperimetric condition (1.4). As is shown in /12/, the linear dependence of the vectors

$$\xi_0 f^\circ + \xi_1 f_1^{\,1} + \xi_2 f_2^{\,2} + \xi_3 f_1^{\,2} = 0 \tag{2.6}$$

follows from (2.5) and (1.4), where ξ_i $(i = 0,\ 1,\ 2,\ 3)$ are constants satisfying the inequality

$$\xi_1\xi_2 \geqslant {}^1/_4\,\xi_3^2 \tag{2.7}$$

Remark. Condition (2.7) is the necessary condition for the maximum of a minimum double eigenvalue if the rank of the system of vectors $f^\circ, f_1^{\,1}, f_2^{\,1}, f_2^{\,2}$ equals 3. If it equals 2 then we select the vectors $f^\circ, f_2^{\,1}$, say, as basis and expand the vectors $f_1^{\,1}, f_2^{\,2}$ in them: $f_1^{\,1} = \alpha_0 f^\circ + \alpha_1 f_2^{\,1}$, $f_2^{\,2} = \beta_0 f^\circ + \beta_1 f_2^{\,1}$. Substituting these expansions into (2.5), we obtain the following necessary condition instead of (2.7): $1 - \alpha_1\beta_1 \geqslant 0$.

For simplicity, below, we will assume the rank of the vector system f°, $f_1^{\,1}$, $f_2^{\,2}$, $f_2^{\,1}$ equals 3.

If the form (2.5) is strictly positive for all non-zero k satisfying condition (1.4), then $\mu_1\mu_2 < 0$. Therefore, the positive-definiteness of the form (2.5) is a sufficient condition for the optimality of the parameter vector h.

However, non-zero variations of k always exist for parameter-vector dimensions $n > 3$, for which the form (2.3) vanishes. In particular, if inequality (2.7) is strictly satisfied, then $L(h, k) = 0$ if and only if

$$(f_s^l, k) = 0, \ (f^\circ, k) = 0, \ l, \ s = 1, \ 2 \tag{2.8}$$

According to (2.4) and (2.2), it therefore follows that: $\mu_1 = \mu_2 = 0$. We let K denote the set of vectors k satisfying condition (2.8).

Thus, if the necessary conditions for the extremum (2.6) and (2.7) are satisfied, where (2.7) is satisfied with the strict inequality sign, then for all allowable variations $k \not\equiv K$ we have $L(h, k) > 0$ and $\mu_1\mu_2 < 0$. The form $L(h, k) = 0$ only for $k \in K$. The values $\mu_1 = \mu_2 = 0$. correspond to this case. Therefore, for the variations $k \in K$ the double eigenvalue λ_1 is not split to a first approximation and the question of the extremality of the vector $h\ (n > 3)$ can be solved by relying on the second variations of the double eigenvalue λ_1 in the set of variations $k \in K$.

We first determine the vector v^1. To do this, we represent v^1 in the form of the expansion (1.11) in eigenvectors, we replace u^1 by $u^\circ = \gamma_1 u^1 + \gamma_2 u^2$ in (1.6), and we multiply (1.6) scalarly by u^i $(i = 3, 4, \ldots, m)$. Hence, taking $\mu = 0$ into account, we find the coefficients c_l. We finally obtain

$$v^1 = c_1 u^1 + c_2 u^2 - \sum_{l=3}^{m} \frac{(C_1(h, k)\, u^\circ, u^l)}{\lambda_l - \lambda_1} u^l \tag{2.9}$$

where the constants c_1 and c_2 are determined from the normalization condition and are not essential to the subsequent computations.

We replace u^1 in (1.7) by $u^\circ = \gamma_1 u^1 + \gamma_2 u^2$ and multiply it successively by u^1 and u^2. Taking account of (2.9) and the condition $\mu = 0$, we obtain a system of linear homogeneous equations in γ_1 and γ_2. From the condition that the determinant of this system equal zero we obtain a quadratic equation in η

$$\eta^2 - \eta(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}^2 = 0 \tag{2.10}$$

$$\beta_{ij} = (C_2(h, k)\, u^i, u^j) - \sum_{l=3}^{m} \frac{(C_1(h, k)\, u^i, u^l)\,(C_1(h, k)\, u^j, u^l)}{\lambda_l - \lambda_1}, \quad i, j = 1, 2$$

As before, the matrices C_1 and C_2 are determined by relations (1.8). The roots of (2.10) are real because of the symmetry of the coefficients β_{ij}.

Therefore, the second varations of the double eigenvalue λ_1 in the class of variations $k \in K$ are determined from (2.10).

Let us formulate the sufficient conditions for the extremum by assuming that the dimensionality of the vector h is greater than 3 (the case $n \leqslant 3$ is considered in /12/).

Assertion 2. Let the following conditions be satisfied: a) a double minimum eigenvalue λ_1 corresponds to the vector h satisfying condition (1.3); b) the necessary extremum conditions (2.6), (2.7) are satisfied, where (2.7) is satisfied with the strict inequality sign. Then the vector h reaches a local maximum of the minimum eigenvalue of problem (1.1) under the isoperimetric condition (1.3) if the minimum of the roots of (2.10) in the class of variations $k \in K$ is less than zero $\eta = \min(\eta_1, \eta_2) < 0, \ k \in K$.

Proof. Because of the conditions a) and b), the form (2.5) is non-negative and equals zero only for variations k satisfying condition (2.8), i.e., for $k \in K$. Hence, it follows that $\mu_1\mu_2 < 0$ for $k \notin K$ and $\mu_1 = \mu_2 = 0$ for $k \in K$. In the former case this means $\min(\mu_1, \mu_2) < 0$, while in the latter case the splitting of the double eigenvalue is determined by the second variations of η_1 and η_2. The set of conditions $\min(\mu_1, \mu_2) < 0, \ k \notin K$ and $\min(\eta_1, \eta_2) < 0, \ k \in K$ denotes the presence of a local maximum.

Note that the condition $\eta = \min(\eta_1, \eta_2) < 0$ is known to be satisfied if

$$\eta_1 + \eta_2 = \beta_{11} + \beta_{22} < 0.$$

Using the expresion for β_{ij} in (2.10) and the notation (2.3), we have

$$\eta_1 + \eta_2 = (D_1(h)\, k, k) - \sum_{l=3}^{m} \frac{(f_1^l, k)^2 + (f_2^l, k)^2}{\lambda_l - \lambda_1} \tag{2.11}$$

where $D_1(h)$ is a matrix with the components

$$d_{st}^1 = \frac{1}{2} \sum_{i,\, j=1}^{m} (u_i^1 u_j^1 + u_i^2 u_j^2) \left(\frac{\partial^2 a_{ij}}{\partial h_s \partial h_t} - \lambda_1 \frac{\partial^2 b_{ij}}{\partial h_s \partial h_t} \right), \quad s, t = 1, 2, \ldots, n$$

Expression (2.11) is a quadratic form in the vector component k. The deduction that the vector h satisfying condition (1.3) and the strengthened conditions (2.6) and (2.7) realizes a local maximum of the minimum double eigenvalue if the quadratic form (2.11) is negative-definite in the set $k \in K$ follows from the proved Assertion 2. The last condition is known to be satisfied if the matrix $D_1(h)$ is negative-definite.

In the case of a linear dependence of the matrices A and B of the problem (1.1) on the vector component h the matrix $D_1 \equiv 0$. In that case, if the rank of the system of vectors $Z_1 \cup Z_2$, where $Z_1 = \{f_1^l\}$, $Z_2 = \{f_2^l\}$ $(l = 1, \ldots, m)$, equals n, then $\eta_1 + \eta_2 < 0$ and the sufficient condition for the extremum is satisfied. The proof is similar to the reasoning in Sect.1.

3. We will examine the infinite-dimensional case in the example of the buckling problem for a thin elastic rod of variable section subjected to a longitudinal force λ. It is assumed that the rod cross-sections are geometrically similar and identically oriented figures. In this case the moment of inertia $I(x) = \alpha h^2(x)$, where $h(x)$ is the cross-sectional area, and α is a constant governed by the section geometry.

The rod deflection function $w(x)$ is determined for buckling from an eigenvalue problem written in dimensionless variables /3/

$$(h^2 w'')'' + \lambda w'' = 0, \quad 0 < x < 1 \tag{3.1}$$

Let us consider two kinds of boundary conditions: "free end-clamping" and "clamping-clamping"

$$(h^2 w'')_{x=0} = [(h^2 w'')' + \lambda w']_{x=0} = 0, \quad w(1) = w'(1) = 0 \tag{3.2}$$

$$w(0) = w'(0) = 0, \quad w(1) = w'(1) = 0 \tag{3.3}$$

For continuous functions $h(x) > 0$, $x \in [0, 1]$, it is known /13/ that the eigenvalue problem (3.1), (3.2) or (3.1),(3.3) possesses a discrete spectrum $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ with eigenfunctions $w_i(x)$ satisfying the orthogonality condition (δ_{ij} is the Kronecker delta)

$$\int_0^1 w_i' w_j' \, dx = \delta_{ij}, \quad i, j = 1, 2, \ldots$$

If the function $h(x)$ vanishes on the boundary of the segment $[0, 1]$, at the point $x = 0$, say, then for positive-definiteness of the eigenvalue problem it is sufficient to require that the following integral should be bounded /13/:

$$\int_0^1 dx \int_x^1 h^{-2}(s) \, ds < \infty \tag{3.4}$$

We shall assume this condition to be satisfied.

The eigenvalue problem (3.1), (3.2) or (3.1), (3.3) can be reduced to a problem with a second-order differential operator. To do this, the substitution $y = h^2 w''$ is used /14/. Consequently, we obtain in palce of (3.1)

$$y'' + \lambda h^{-2} y = 0, \quad 0 < x < 1 \tag{3.5}$$

We obtain the boundary conditions for the function y by double integration of the equation $y'' + \lambda w'' = 0$ using the boudary conditions (3.2) or (3.3). We hence have /14/

$$y(0) = 0, \quad y'(1) = (0) \tag{3.6}$$

$$y'(0) = y'(1), \quad y(1) = y(0) + y'(0) \tag{3.7}$$

An eigenfunction y_i with the same eigenvalue λ_i of the problem (3.5), (3.6) or (3.5), (3.7) corresponds uniquely to every eigenfunction $w_i(x)$ of the problem (3.1), (3.2) or (3.1), (3.3) corresponding to the eigenvalue λ_i, and conversely, the eigenfunction w_i corresponding to the same λ_i corresponds uniquely to each eignefunction y_i corresponding to the eigenvalue $\lambda_i \neq 0$. It hence follows that the spectra of problems (3.5), (3.6) and (3.5), (3.7) are non-negative since all the eigenvalues are positive in problems (3.1), (3.2), and (3.1), (3.3).

It can be established by direct substitution that there are no zero eigenvalues in problem (3.5), (3.6); consequently, the spectrum in problems (3.1), (3.2) and (3.5), (3.6) is completely identical.

There is a double zeroth eigenvalue $\lambda_0^1 = \lambda_0^2 = 0$ in problem (3.5), (3.7) /15, 16/ to which two linearly independent eigenfunctions 1 and x correspond, and which occur because of the passage to a problem with a second-order operator.

The eigenfunctions of the problem (3.5), (3.6) and (3.5), (3.7) can be orthonormalized

$$\int_0^1 y_i y_j h^{-2} \, dx = \delta_{ij} \tag{3.8}$$

The eigenfunction system $\{y_i\}$ is complete in the space L_2^h of square-integrable functions with weight h^{-2}. The norm in this space is determined by the expression

$$\|y\|^2 = \int_0^1 h^{-2}y^2\,dx \tag{3.9}$$

We now turn to the following optimization problem: it is required to find the continuous function $h(x) \geqslant 0$ that maximizes the first (non-zero) eigenvalue λ_1 of problem (3.5), (3.6) or (3.5), (3.7) for a constraint on the rod volume

$$\int_0^1 h\,dx = 1 \tag{3.10}$$

Lagrange formulated this problem, which has been examined in many papers /1–5, 14–18/. It is known /15, 16/ that the eigenvalue λ_1 in problem (3.5), (3.6) is always simple, whereas it may turn out to be double in problem (3.5), (3.7). We hence consider these cases separately.

4. We investigate the optimization problem with boundary conditions (3.6). It can be proved that the solution $h(x)$ satisfying first-order necessary conditions realizes a local maximum of the minimum eigenvalue of problem (3.5), (3.6) under the constraint (3.10).

For the proof we use the perturbation method and we obtain expressions for the first and second variations of a simple eigenvalue λ_1. We give the function $h(x)$, which could have the extremum, an increment $\varepsilon\delta h(x)$, where ε is a small positive number. Consequently, the first eigenvalue λ_1 and the corresponding eigenfunction $y_1(x)$ receive the increments /11/

$$\lambda = \lambda_1 + \varepsilon\mu + \varepsilon^2\eta + \ldots,\quad y(x) = y_1(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \ldots$$

Substituting these expansions into (3.5) and (3.6), and collecting terms of identical powers of ε we obtain

$$\begin{aligned} &y_1'' + \lambda_1 h^{-2} y_1 = 0 \\ &v_1'' + \lambda_1 h^{-2} v_1 = 2\lambda_1 h^{-3}\delta h y_1 - \mu h^{-2} y_1 \\ &v_2'' + \lambda_1 h^{-2} v_2 = -6\lambda_1 h^{-4}(\delta h)^2 y_1 + 2\lambda_1 h^{-3}\delta h v_1 + 2h^{-3}\mu\delta h y_1 - \mu v_1 h^{-2} - \eta h^{-2} y_1 \\ &y_1(0) = y_1'(1) = 0,\quad v_i(0) = v_i'(1) = 0,\quad i = 1, 2 \end{aligned} \tag{4.1}$$

We mutliply the second equation in (4.1) by $y_1(x)$ and integrate the result between 0 and 1. Later, integrating by parts and using the first equation in (4.1) and condition (3.8), we obtain an expression for the first variation

$$\mu = 2\lambda_1 \int_0^1 h^{-3} y_1^2 \delta h\,dx \quad \left(\int_0^1 \delta h\,dx = 0\right)$$

where the condition in parentheses follows from the constraint (3.10). Because of the arbitrariness of the variation δh, we hence obtain the necessary optimality condition

$$y_1^2(x) h^{-3}(x) = \varkappa^2,\quad \varkappa = \text{const} \tag{4.2}$$

Equations (3.5), (3.6), (4.2) and conditions (3.8), (3.10) are for determining the unknown functions $y_1(x), h(x)$ and the values $\lambda_1, \varkappa$. The analytic solution of these equations was first obtained by Clausen /17/, see /18, 14/ also. The function $h(x)$ vanishes at the point $x = 0$, where we have $h(x) \sim x^{2/3}$ in the neighbourhood of $x = 0$, consequently, condition (3.4) is satisfied.

We will now derive on expression for the second variation η of the eigenvalue λ_1. For this we first represent the function $v_1(x)$ as an expansion in eigenfunctions $y_i(x)$, i.e., $v_1(x) = c_1 y_1(x) + \ldots$. The coefficients $c_l\ (l = 1, 2, \ldots)$ are found from the second equation in (4.1) by multiplying it by $y_i(x)\ (i = 2, 3, \ldots)$, integrating between 0 and 1 and using (3.8) and the first equation in (4.1). We consequently have

$$v_1(x) = c_1 y_1(x) - 2\lambda_1 \sum_{l=2}^{\infty} (\lambda_l - \lambda_1)^{-1} g_{l1} y_l(x) \tag{4.3}$$

$$g_{ls} = \int_0^1 h^{-3} y_s y_l \delta h\,dx \quad (l, s = 1, 2, \ldots)$$

The coefficient c_1 is found from the normalization condition and does not affect the subsequent computations.

By using (4.3) and taking account of the condition $\mu = 0$ we obtain an expression for the second variation from the third equation of (4.1)

$$\eta = -6\lambda_1 \int_0^1 h^{-4} y_1^2 (\delta h)^2\,dx - 4\lambda_1^2 \sum_{l=2}^{\infty} (\lambda_l - \lambda_1)^{-1} g_{l1}^2 \tag{4.4}$$

Since $\lambda_l > \lambda_1 > 0\ (l = 2, 3, \ldots)$, the second term in (4.4) is non-positive. Using condition (4.2), we obtain the estimate

$$\eta \leqslant -6\lambda_1 \varkappa^2 \int_0^1 h^{-1} (\delta h)^2 \, dx < 0 \qquad (4.5)$$

The integral in (4.5) is the square of the norm $\| \delta h \|_p^2$ in the space of square-integrable functions with weight $p = h^{-1}(x)$. The negativity of the second variation denotes that the function $h(x)$ satisfying the necessary condition for an extremum realizes the local maximum of λ_1 under the condition (3.10), which it was required to prove.

Another proof of the optimality of λ_1, based on application of the Hölder inequality, is given in /14/. Proof of the optimality of solutions satisfying the necessary conditions for an extremum are presented in /19/ for sandwich structures $(I(x) = \alpha h(x))$.

5. We consider the optimization problem formulated in Sect.3 with the boundary conditions (3.7). It has been shown /15, 16/ that the solution of the optimization problem can be characterized by just the double eigenvalue $0 < \lambda_1 = \lambda_2 < \lambda_3 \leqslant \lambda_4 \ldots$ We will prove that the function $h(x)$ satisfying the necessary conditions of the double λ_1 realizes a local maximum of λ_1 under the condition (3.10).

Proof. It was shown in Sect.3 that problem (3.5), (3.7) has a double zeroth eigenvalue $\lambda^1{}_0 = \lambda^2{}_0 = 0$ and corresponding linear eigenfunctions $y^1{}_0 = ax + b$, $y^2{}_0 = cx + d$ for any $h(x)$. It is assumed that all the eigenfunctions are orthonormalized

$$\int_0^1 y_0{}^s y_0{}^l h^{-2} \, dx = \delta_{sl}, \quad s, l = 1, 2; \quad \int_0^1 y_i y_j h^{-2} \, dx = \delta_{ij}, \quad i, j = 1, 2, \ldots \qquad (5.1)$$

We give the function $h(x)$, which could have an extremum, an increment $\varepsilon\delta h(x)$ and we apply the perturbation method. Exactly as in Sect.2, the first variations μ_1 and μ_2 of the double λ_1 are found from the solution of the quadratic equation

$$\mu^2 - \mu(\beta_{11} + \beta_{22}) + \beta_{11}\beta_{22} - \beta_{12}{}^2 = 0 \qquad (5.2)$$

$$\beta_{ij} = 2\lambda_1 \int_0^1 h^{-3} y_i y_j \delta h \, dx, \quad i, j = 1, 2$$

If $h(x)$ reaches the maximum of the double λ_1, then $\mu_1\mu_2 = \beta_{11}\beta_{22} - \beta_{12}{}^2 \leqslant 0$ is necessary for any variations δh satisfying (3.10). Hence follows the linear dependence of the functions /12/

$$f_0 = 1, \; f_1 = h^{-3} y_1{}^2, \; f_2 = h^{-3} y_2{}^2, \; f_3 = h^{-3} y_1 y_2 \qquad (5.3)$$

$$\xi_0 f_0 + \xi_1 f_1 + \xi_2 f_2 + \xi_3 f_3 = 0$$

with coefficient ξ_i satisfying the inequality (2.7).

The functions h, y_1, y_2 and the constants ξ_i, $i = 0, 1, 2, 3$, realizing the extremum of the optimization problem can be found from system (3.5), (3.7) written for y_1, λ_1 and $y_2, \lambda_2 = \lambda_1$ and the relationships (5.3), (2.7), and (3.10). The analytic solution of the system of these equations was obtained in /15, 16/, where it is shown that the function $h(x) > 0$, $x \in [0, 1]$.

If inequality (2.7) is satisfied strictly, then the form $\beta_{11}\beta_{22} - \beta_{12}{}^2$ equals zero if and only if the variations δh satisfy the conditions $\beta_{11} = \beta_{22} = \beta_{12} = 0$ and the conditions (3.10), i.e.

$$\int_0^1 f_i \delta h \, dx = 0; \quad i = 0, 1, 2, 3 \qquad (5.5)$$

We let Δ denote the class of variations δh satisfying (5.5). It follows from (5.2) that $\mu_1 = \mu_2 = 0$ if $\delta h \in \Delta$. Otherwise $(\delta h \notin \Delta)$ $\mu_1\mu_2 < 0$. This means that these variations $(\delta h \notin \Delta)$ reach the maximum of double λ_1 since here $\min(\mu_1, \mu_2) < 0$.

Thus, to prove the optimality of $h(x)$, the second variations of the double λ_1 should be examined in the class $\delta h \in \Delta$ and it is seen that at least one of these variations is strictly negative.

Exactly as in Sect.2, the second variations of η_1 and η_2 are determined for $\delta h \in \Delta$ from the solution of the quadratic equation

$$\eta^2 - \eta(\gamma_{11} + \gamma_{22}) + \gamma_{11}\gamma_{22} - \gamma_{12}{}^2 = 0 \qquad (5.6)$$

The minimum root of (5.6) will be negative if $\eta_1 + \eta_2 < 0$. The sum $\eta_1 + \eta_2$ is represented in the class of variations $\delta h \in \Delta$ by the expression

$$\eta_1 + \eta_2 = K_1 + K_2 \qquad (5.7)$$

$$K_1 = \lambda_1 \sum_{i=1}^{2} \left[-6 \int_0^1 y_i{}^2 h^{-4} (\delta h)^2 \, dx + 4 \sum_{j=1}^{2} \left(\int_0^1 y_0{}^j y_i h^{-3} \delta h \, dx \right)^2 \right]$$

$$K_2 = -4\lambda_1{}^2 \sum_{l=3}^{\infty} \sum_{i=1}^{2} (\lambda_l - \lambda_1)^{-1} g_{il}{}^2$$

Because $\lambda_l > \lambda_1$ $(l = 3, 4, \ldots)$ $K_2 \leqslant 0$. Let us estimate K_1. For this we introduce the auxiliary functions $\psi_1(x) = y_1 h^{-1} \delta h$ and $\psi_2(x) = y_2 h^{-1} \delta h$. Because of the continuity of the functions

$y_1, y_2, \delta h, h \geqslant \delta > 0$, the functions ψ_1, ψ_2 belong to the space L_2^h of square-summable functions with weight h^{-2}

$$\|\psi_i\|^2 = \int_0^1 y_i^2 h^{-4} (\delta h)^2 \, dx < +\infty \tag{5.8}$$

The system of eigenfunctions $y_i(x)$ of problem (3.5), (3.7) is complete in L_2^h; we consequently expand $\psi_i(x)$ in it

$$\psi_i(x) = d_0^{i,1} y_0^1(x) + d_0^{i,2} y_0^2(x) + \sum_{l=1}^{\infty} d_l^i y_l(x) \quad (i = 1, 2)$$

where

$$d_l^{\,i} = \int_0^{1} \psi_i y_l h^{-2} \, dx = \int_0^1 y_i y_l h^{-3} \delta h \, dx \quad (l = 1, 2, \ldots) \tag{5.9}$$

$$d_0^{\,ij} = \int_0^{1} y_0^j \psi_i h^{-2} \, dx = \int_0^1 y_i y_0^j h^{-3} \delta h \, dx \quad (i, j = 1, 2)$$

We note that the coefficients d_l^1, d_l^2 $(l = 1, 2)$ equal zero because of condition (5.5). Because of the orthonormality of the system $\{y_i(x)\}$ we have

$$\|\psi_i\|^2 = (d_0^{i,1})^2 + (d_0^{i,2})^2 + \sum_{l=3}^{\infty} (d_l^{\,i})^2 \quad (i = 1, 2) \tag{5.10}$$

The estimate

$$\eta_1 + \eta_2 \leqslant K_1 \leqslant -2\lambda_1 (\|\psi_1\|^2 + \|\psi_2\|^2) = -2\lambda_1 \int_0^1 (y_1^2 + y_2^2) h^{-4} (\delta h)^2 \, dx < 0 \tag{5.11}$$

follows from (5.7)–(5.10).

The integral in the last expression is the square of the norm $\|\delta h\|_p^2$ in the space of square-integrable functions with weight $p = (y_1^2 + y_2^2) h^{-4}$.

Thus, we have $\eta_1 + \eta_2 < 0$ from (5.11), therefore, $\min(\eta_1, \eta_2) < 0$ in the class $\delta h \in \Delta$, which it was required to prove.

We note that these proofs can be executed analogously for other cases of the dependence between the moment of inertia and the cross-sectional area, $I(x) = \alpha h^3(x)$, say, as well as for other (selfadjoint) boundary conditions of rod clamping).

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